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Einstein's Theory of Quantum Radiation

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The stationary probability distribution of the one-step process corresponding to Einstein's theory of absorption and emission of radiation is derived. Gauss' principle is used to identify the entropy, and the second law gives the dynamical equilibrium condition or Planck's radiation law. This condition is in disagreement with Einstein's criterion of dynamical equilibrium. The physical consequences of the new condition are investigated.

1. INTRODUCTION

Einstein (1917) had the great physical intuition of attempting a derivation of Planck's radiation law from a physical mechanism of absorption and emission of radiation. This avoided any mention of the connection between entropy and the logarithm of the negative binomial coefficient, which was rationalized by Ehrenfest and Kamerlingh Onnes (1914) as the number of ways of permutating the number of "energy grades" (quanta) and "dividers" separating the individual oscillators.

Crucial to Einstein's argument were the assumptions of a dynamical equilibrium between the rates of absorption and emission of radiation and that the molecules satisfy Maxwell-Boltzmann statistics. However, Einstein had three unknown coefficients, the absorption, stimulated, and spontaneous emission coefficients, with only a single relation, representating dynamical equilibrium, among them. In order to eliminate one of the unknowns, Einstein took the asymptotic high-temperature limit, from which he concluded that the coefficients of absorption and stimulated emission are equal in the case that the statistical weights (which are numbers, independent of the temperature) of the two states are equal. The ratio of the two remaining coefficients could then be identified with the corresponding term in Planck's law, representing the number of oscillators between the frequencies ν and $\nu+d\nu$.

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Einstein avoided both Boltzmann's combinatorial argument relating the entropy to the "thermodynamic" probability and the second law by postulating a dynamical equilibrium between the rates of absorption and emission of radiation at equilibrium. However, Einstein's condition of dynamical equilibrium must necessarily be compatible with the second law. In fact, Boltzmann's procedure can be circumvented, at the cost of relinquishing the assumption of *a priori* probability, by solving the one-step master equation for the stationary probability distribution and showing that this distribution satisfies Gauss' principle, which chooses that distribution for which the most likely and average value coincide. The potential appearing in Gauss' expression for the probability distribution is the entropy of the process (Lavenda, 1988). The second law then establishes the criterion for dynamical equilibrium which is comparable to Einstein's condition. Noticing a discrepancy between the two relations, I derive the diffusion coefficient entering into the dynamic equilibrium between the (unsystematic) force of the radiation pressure on the electron which causes it to accelerate and (systematic) force arising from the reaction of the radiation field tending to slow the particle down because the energy radiated by a moving charge causes a decrease in its kinetic energy. The motion is influenced by the radiation field and conservation of energy dictates that we have to consider the reaction of the field on the moving charge (e.g., Heitler, 1954).

Einstein's strong attraction to Brownian motion (Einstein, 1905) led Einstein and Hopf (1910) to apply the same type of dynamical equilibrium that exists between the osmotic pressure forces and an arbitrary external force that resists the motion of the Brownian particle. With the condition of dynamic equilibrium formulated in velocity rather than configuration space, Einstein and Hopf concluded that the momentum fluctuations calculated using the equipartition law were far too small to account for the observed fluctuations, especially at high frequencies. The same type of equipartition law was used by Planck (1899) in equating the rates at which energy is emitted and absorbed by an oscillator under equilibrium conditions. The difficulty occurs in the region of high frequencies or low temperatures (Einstein, 1906), where considerations based on the law of equipartition that are valid at low frequencies and moderate temperatures predict that the rate of absorption, as well as the intensity of the electromagnetic field, should tend to zero (i.e., Wien's law) with the temperature. The aim of this paper is to show that it is precisely the correction to Einstein's condition of dynamic equilibrium that leads to a zero-point energy implying a finite rate of energy absorption and a finite pressure at very low temperatures or high frequencies. Curiously enough, the same zero-point energy was proposed by Einstein and Stern (1913) in their arguments for the existence of molecular agitation at zero K.

2. DERIVATION OF THE PROBABILITY DISTRIBUTION

Einstein (1917) derived Planck's radiation law by considering a physical mechanism of emission and absorption of radiation by matter. I will now derive the probability distribution which gives rise to that law and in so doing show that the assumption of *a priori* probabilities is untenable. The entropy will subsequently be identified using Gauss' principle for the derivation of the normal error law (Lavenda, 1988).

Following Einstein (1917), we consider the simplified case of absorption and emission of radiation between two levels. The rate of absorption of radiation is proportional to the number of photons, $\vartheta_r(n) = \alpha n$, while the rates of stimulated and spontaneous emission are given by $\vartheta_g(n) = \beta n + \gamma$, where the term independent of n describes spontaneous emission. The master equation is

$$\dot{f}(n) = \{(\mathcal{T} - 1)\alpha n + (\mathcal{T}^{-1} - 1)(\beta n + \gamma)\} f(n) \tag{1}$$

where the "step operators" $\mathcal{T}$ and $\mathcal{T}^{-1}$ act on an arbitrary function $\varphi(n)$ to give $\mathcal{T}\varphi(n) = \varphi(n+1)$ and $\mathcal{T}^{-1}\varphi(n) = \varphi(n-1)$, respectively (e.g., van Kampen, 1981). The stationary solution f^s of the master equation (1) satisfies

$$\alpha n f^s(n) = [\beta(n-1) + \gamma] f^s(n-1), \tag{2}$$

provided there is no net probability flow from $n \to n-1$ (i.e., detailed balance). The stationary solution has the form of a negative binomial distribution

$$f^s(n; m, p) = \binom{m+n+1}{n} p^m q^n \tag{3}$$

Introducing the distribution (3) into condition (2) yields $\alpha(m+n-1)q = \beta(n-1) + \gamma$, which must be satisfied identically for all values of n. This requires that $q = \beta/\alpha$ and $m = \gamma/\beta$. The former implies that $\alpha > \beta$, since $p = (\alpha - \beta)/\alpha$.

Two points are worth noting; first, according the traditional interpretation (Ehrenfest and Kamerlingh Onnes, 1914), m represents the number of "cells" or "oscillators," which must necessarily be a positive integer. We have set this equal to the ratio of rate constants of spontaneous and stimulated emission, which need not be an integer. In fact, the normalizing condition

$$\sum_{n=0}^{\infty} f^s(n; m, p) = 1 \tag{4}$$

holds for *any positive m*. Second, existence of an asymptotic stationary solution

$$\bar{n}^s = \frac{\gamma}{\alpha - \beta} \tag{5}$$

to the average equation of motion

$$\dot{\bar{n}} = -\alpha\bar{n} + \beta\bar{n} + \gamma \tag{6}$$

depends upon the same condition that $p > 0$, namely $\alpha > \beta$, which requires the rate of absorption of radiation to be greater than the rate of stimulated emission. The three relations for m, p, and $\bar{n}^s$ give $\bar{n}^s/m = q/p$, from which it follows that $p = m/(m + \bar{n}^s)$ and $q = \bar{n}^s/(m + \bar{n}^s)$.

3. GAUSS' PRINCIPLE AND THE SECOND LAW

Gauss (1963), in deriving the normal law of error, assumed that the arithmetic mean of the measurements is the most probable value of the quantity measured. Since the normal law is symmetrical (positive and negative errors of the same absolute magnitude are equally likely), the mode and the mean of the distribution are identical. I further assume that the arithmetic mean of the measurements coincides with the mean of the distribution, which is rigorously so when the number of measurements increases without limit. By Gauss' principle I shall mean that the distribution should be chosen such that the expected value is the most likely value.

Let $f^s(n; \bar{n}^s)$ denote the probability that a measurement will give the value n whose true value is known to be $\bar{n}^s$, its average value, which according to assumption coincides with its most probable value. In order to determine the law of error, observe that (Keynes, 1921) $(1/f^s)\,\partial f^s/\partial\bar{n}^s = 0$ must be equivalent to $n - \bar{n}^s = 0$. Therefore,

$$\frac{\partial \ln f}{\partial \bar{n}^s} = \varphi(\bar{n}^s)(n - \bar{n}^s) \tag{7}$$

where $\varphi(\bar{n}^s)$ is some function that is independent of n. Integrating (7) gives

$$\ln f^s(n; \bar{n}^s) = -(n - \bar{n}^s)\frac{\partial \mathcal{S}}{\partial \bar{n}^s} - \mathcal{S}(\bar{n}^s) + \Sigma(n) \tag{8}$$

where $\Sigma(n)$ is some function independent of $\bar{n}^s$. I have set $\varphi = -\partial^2\mathcal{S}/\partial\bar{n}^{s2}$, where $\mathcal{S}$ is some "potential" function whose existence can be rigorously demonstrated in the multidimensional case where the exactness conditions must be satisfied.

I now show that the negative binomial distribution (3) is a law of error leading to the average value as the most probable value. Taking the logarithm of (3) and expressing it in the form of (8), let us identify the potential as

$$\mathcal{S}(\bar{n}^s) = (m+\bar{n}^s)\ln(\bar{n}^s+m) - \bar{n}^s \ln \bar{n}^s - m\ln m \tag{9}$$

which is easily recognized as the (statistical) entropy. For n and m sufficiently large to justify the use of Stirling's approximation, the last term in expression (8) becomes

$$\Sigma(n) = (m+n)\ln(m+n) - n\ln n - m\ln m \tag{10}$$

which I have referred to as the "stochastic" entropy (Lavenda, 1988). Furthermore, the expectation value $\bar{n}^s$ maximizes the distribution since

$$\left(\frac{\partial^2 \mathcal{S}}{\partial n^2}\right)_{n=\bar{n}^s} = -\frac{m}{\bar{n}^s(\bar{n}^s+m)} = -\left(\frac{mq}{p^2}\right)^{-1} = -\sigma_\infty^{-2} < 0$$

where σ_∞^2 is the variance of the negative binomial distribution.

In addition to the average number of particles, the (thermodynamic) entropy is a function of the volume V and average energy $\bar{\mathscr{E}}$, which, if the particle energy is ε, is given by $\bar{\mathscr{E}} = \bar{n}^s\varepsilon$. Then, at constant volume,

$$\left(\frac{\partial \mathcal{S}}{\partial \bar{n}^s}\right)_V = \left(\frac{\partial \mathcal{S}}{\partial \bar{n}^s}\right)_{\bar{\mathscr{E}},V} + \left(\frac{\partial \mathcal{S}}{\partial \bar{\mathscr{E}}}\right)_{\bar{n}^s,V} \frac{d\bar{\mathscr{E}}}{d\bar{n}^s} = \frac{-\mu+\varepsilon}{T} \tag{11}$$

where μ is the chemical potential and the absolute temperature T is measured in energy units. If the system is in thermal equilibrium with the heat bath at temperature T, then the derivative of the thermodynamic entropy, given in (11), will be equal to the derivative of the statistical entropy,

$$\frac{\partial \mathcal{S}}{\partial \bar{n}^s} = \ln\left(\frac{m+\bar{n}^s}{\bar{n}^s}\right) \tag{12}$$

which gives precisely the Bose-Einstein distribution

$$\bar{n}^s = \frac{m}{e^{(\varepsilon-\mu)/T}-1} \tag{13}$$

The Bose-Einstein distribution (13) can be expressed as

$$\frac{\beta}{\alpha} = e^{-(\varepsilon-\mu)/T} \tag{14}$$

which corresponds, in Einstein's terminlogy, to a condition of "dynamical equilibrium."

Since the number of photons is not conserved, we must set $\mu = 0$, and with $\varepsilon = h\nu$ the dynamical equilibrium condition (14) becomes

$$\frac{\beta}{\alpha} = e^{-h\nu/T} \tag{15}$$

However, this dynamical equilibrium condition is at variance with Einstein's (1917) condition

$$\alpha \bar{n}^s e^{-\varepsilon_i/T} = (\beta \bar{n}^s + \gamma)\, e^{-\varepsilon_j/T} \tag{16}$$

equating the rates of absorption and emission of radiation between two states of equal statistical weight with energies ε_i and ε_j such that $\varepsilon_i < \varepsilon_j$ and $\varepsilon_j - \varepsilon_i = h\nu$. Equation (16) is a single condition with three unknowns. In order to eliminate one of the unknowns so that the remaining two would only appear in a ratio, Einstein took the limit as $T \to \infty$. Since $\bar{n}^s$ goes to infinity with T, this gave the condition $\alpha = \beta$ in this limit, provided the numerical statistical weights for the two states are equal. Assuming that this equivalence would hold at *any* temperature, Einstein introduced it into his dynamical equilibrium condition (16) to obtain

$$\frac{1 + \bar{n}^s/m}{\bar{n}^s/m} = e^{h\nu/T} \tag{17}$$

which is precisely Planck's distribution law where $m = \gamma/\alpha$. However, there is really no justification for introducing the asymptotic equivalence between the coefficients of absorption and induced emission that has been obtained from (16) in the limit as $T \to \infty$ back into the same relation which is valid for finite T. Rather, the stationary state expression (5) together with (15), which has been derived from the second law, is sufficient to establish Planck's radiation law.

The first relation between the Einstein coefficients that I have found is given by (15) and it is this relation which is at variance with the one found by Einstein except in the limit of low frequencies or high temperatures. The second relation, $\gamma/\beta = m$, becomes $\gamma/\beta = 8\pi\nu^2 V/c^3$ when m is identified as the number of oscillators between frequencies ν and $\nu + d\nu$. According to Einstein, γ/β should be $h\nu$ times my expression and $\beta = \alpha$ in the case where the statistical weights of the two states are equal. Let us observe that only ratios can be obtained from thermodynamic arguments and those ratios should be independent of Planck's constant since they must be identical to the corresponding expressions of classical electrodynamics.

With my values of the coefficients, the entropy (9), which is the total entropy of m oscillators at the equilibrium temperature T and in the frequency range $d\nu$, is

$$\begin{aligned}\mathcal{S}(\nu, T) &= \frac{\alpha}{\alpha - \beta} \ln\left(\frac{\alpha}{\alpha - \beta}\right) - \frac{\beta}{\alpha - \beta} \ln\left(\frac{\beta}{\alpha - \beta}\right) \\ &= \frac{\gamma}{\beta}\left[\frac{h\nu/T}{e^{h\nu/T} - 1} - \ln(1 - e^{-h\nu/T})\right] \end{aligned} \tag{18}$$

The average energy per unit volume in the same frequency range is

$$\bar{\mathscr{E}}(\nu) = \bar{n}^s h\nu = \frac{\gamma}{\beta}\frac{h\nu}{e^{h\nu/T}-1} \tag{19}$$

so that the difference between (18) and T^{-1} times (19) is

$$\ln \mathscr{Q}(\nu, T) = -\frac{\gamma}{\beta}\ln(1-e^{-h\nu/T}) \tag{20}$$

where $\mathscr{Q}$ is the partition function.

4. DYNAMICAL EQUILIBRIUM AND THE DIFFUSION COEFFICIENT

The equation of motion (6) is the average of the stochastic differential equation

$$dn = \{\gamma - (\alpha - \beta)n\}\, dt + dW \tag{21}$$

where W is a Brownian motion, with zero mean and variance $\overline{dW^2} = D\, dt$. The Brownian motion accounts for the statistical aspects of the radiation pressure and D is the diffusion coefficient which I seek to determine. Since the variance

$$\sigma^2 = \overline{(n-\bar{n})^2}$$

I can employ equation (21) to obtain its equation of motion as

$$\frac{1}{2}\frac{d\sigma^2}{dt} = -(\alpha-\beta)\sigma^2 + D \tag{22}$$

In the asymptotic limit as $t \to \infty$ we get $D = (\alpha - \beta)\sigma^2_\infty(\bar{n}^s)$, where the asymptotic value of the variance is [cf. the expression following (10)]

$$\sigma^2_\infty(\bar{n}^s) = mq/p^2 = \bar{n}^s(\bar{n}^s/m+1) \tag{23}$$

It will be appreciated that

$$D = \alpha\bar{n}^s = \frac{\gamma}{1-e^{-h\nu/T}} \tag{24}$$

is the fluctuation-dissipation theorem for Gaussian Markov processes (e.g., Lavenda, 1985). At low frequencies, the diffusion coefficient (24) becomes

$$D = \gamma T/h\nu \tag{25}$$

Since the fluctuations are now classical, equation (25) must be independent of Planck's constant. According to classical electrodynamics, the linear harmonic oscillator equation, acting as a simple model for a light source, with a reaction force is (Heitler, 1954)

$$M\ddot{x} = -M\omega^2 x + \frac{2e^2}{3c^3}\dddot{x} \tag{26}$$

where e is the electric charge, M is the mass of the electron, and $\omega = 2\pi\nu$. If the reaction force is small with respect to the elastic force, then, in first approximation, $\dddot{x} \sim -\omega^2\dot{x}$. Thus, radiation damping will be described by the equation of motion $\ddot{x} + \gamma\dot{x} + \omega^2 x = 0$, where $\gamma = 2e^2\omega^2/3Mc^3$ is the linewidth at half maximum, which is just equal to the total spontaneous transition probability per unit time (Heitler, 1954). Inserting this value into the expression for the diffusion coefficient gives

$$D = \frac{2}{3}\zeta\frac{T}{Mc^2}\omega \tag{27}$$

where $\zeta = e^2/\hbar c$ is the fine-structure constant into which Planck's constant has been incorporated.

In order to obtain the value of the coefficient m, Planck (1899) equated the rates at which energy is emitted and absorbed by an oscillator at equilibrium. The rate of energy absorption $\dot{\mathscr{E}}_a \sim (e^2/Mc)\mathscr{I}$, where $\mathscr{I} \sim c\mathbf{E}^2$ is the intensity and $\mathbf{E}$ is the electric field strength. The classical rate of radiation by a harmonic oscillator is $\dot{\mathscr{E}}_e \sim (e^2\omega^2/Mc^3)\bar{\mathscr{E}}/m$, where $\bar{\mathscr{E}}/m = M\omega^2\overline{x^2}$ is the average energy per oscillator. Equating the rates of emission and absorption of radiation gives $\mathbf{E}^2 \sim (\omega^2/c^3)\bar{\mathscr{E}}/m$ and since $\bar{\mathscr{E}} \sim \mathbf{E}^2$, it follows that $m \sim \omega^2/c^3$, which becomes precisely Planck's result when we introduce a factor of $2/\pi$. Since the damping constant $\gamma \sim (e^2\omega^2/Mc^3)$, it follows that $\beta \sim (e^2/M)$ and from (15) that $\alpha \sim (e^2/M)\, e^{h\nu/T}$. Notice that I have used the equipartition law to obtain the functional dependence of m on the frequency. However, in the opposite limit of high frequencies or low temperatures, the adherence to the same relation would predict that the intensity approaches zero and so does the rate of energy absorption.

The total energy radiated per unit time is the average rate at which photons are emitted $\vartheta_g(\bar{n}^s)$ multiplied by the energy per quantum $h\nu$,

$$\dot{\mathscr{E}}_e = (\gamma + \beta\bar{n}^s)h\nu = \gamma h\nu + \beta\bar{\mathscr{E}} = \frac{\gamma h\nu}{1 - e^{-h\nu/T}} \tag{28}$$

In comparison, the average rate of spontaneous emission is

$$\dot{\mathscr{E}}_e = \gamma\left(\frac{\bar{\mathscr{E}}}{m}\right) = \frac{\gamma h\nu}{e^{h\nu/T} - 1} \tag{29}$$

where the first equality is identical to the average monochromatic radiation emitted by an oscillator of classical theory when $\bar{\mathscr{E}}/m$ is identified with twice the average potential energy of the harmonic oscillator (Heitler, 1954). Although both expressions (28) and (29) reduce to

$$\dot{\mathscr{E}}_e = \gamma T \tag{30}$$

in the low-frequency limit, they differ decisively in the high-frequency limit, where (28) becomes

$$\dot{\mathscr{E}}_e = \gamma h\nu \tag{31}$$

while (30) goes over into

$$\dot{\mathscr{E}}_e = \gamma h\nu \, e^{-h\nu/T} \tag{32}$$

which is what would be predicted from Wien's law, and vanishes either in the high-frequency or low-temperature limit.

To obtain the average rate of absorption of energy, we multiply the average rate at which quanta are absorbed, $\vartheta_r(\bar{n}^s)$, by the energy per quantum. This gives

$$\dot{\mathscr{E}}_a = \alpha\bar{\mathscr{E}} = \frac{\gamma h\nu}{1 - e^{-h\nu/T}} \tag{33}$$

in contrast to what Einstein would have predicted,

$$\dot{\mathscr{E}}_a = \beta\bar{\mathscr{E}} = \frac{\gamma h\nu}{e^{h\nu/T} - 1} \tag{34}$$

on the basis of the assumed equivalence of the coefficients of absorption and induced emission. Both expressions (33) and (34) give the equipartition result

$$\dot{\mathscr{E}}_a = \gamma T \tag{35}$$

in the low-frequency limit but differ in the high-frequency limit: whereas (33) goes over into (31), the average energy absorbed per unit time given by (34) vanishes in the limit as $T \to 0$. In this limit all oscillators should be in their ground state, so that there indeed exists the possibility of energy absorption. This is to be attributed to the addition of a zero-point energy, equal to $h\nu$, in expression (28) which appears in the spontaneous emission term. The result was anticipated by Einstein and Stern (1913), who noted that it enabled Planck's radiation law to be derived "without the usual discontinuity assumptions." The existence of such a zero-point energy is due to the thermal equilibrium between absorption and induced emission which is expressed by relation (15).

Following Einstein and Hopf (1910), I consider a dynamical equilibrium between the "damping" force

$$\mathscr{F} = -M\gamma v \tag{36}$$

where v is the velocity of the oscillator in the x direction and an "osmotic" pressure force originating in the statistical fluctuations of the radiation pressure. I am therefore led to the velocity space diffusion equation

$$\frac{\partial f}{\partial t}=-\frac{\partial}{\partial v}\left\{\mathscr{F}-\mathscr{D}\frac{\partial}{\partial v}\right\}f \tag{37}$$

which is to be solved for its stationary probability density f^s satisfying

$$\left\{\mathscr{F}-\mathscr{D}\frac{d}{dv}\right\}f^s(v)=0 \tag{38}$$

under the condition of detailed balance.

Einstein (1917) and Einstein and Hopf (1910) did not question the validity of the Maxwell distribution law since the resulting equipartition law is only applied to *translatory* motion, for which "this law was always brillantly corroborated" (Einstein and Hopf, 1910). With the stationary distribution given by

$$f^s(v)=\frac{1}{(2\pi T/M)^{1/2}}\,e^{-Mv^2/2T} \tag{39}$$

the dynamical equilibrium condition (38) yields

$$\mathscr{D}_{\text{Ein}}=\gamma T \tag{40}$$

Expression (40) is the Einstein diffusion coefficient for Brownian motion (Einstein, 1905) applied to fluctuations in electromagnetic momentum. Einstein and Hopf (1910) concluded that this value is far too small compared with *real* momentum fluctuations, especially when very high frequencies are involved.

According to the equipartition law, T is the average energy per oscillator, $\bar{\bar{\mathscr{E}}}/m$, so that

$$\mathscr{D}_{\text{Ein}}=\beta\bar{\bar{\mathscr{E}}} \tag{41}$$

since $\gamma/m=\beta$. However, on the basis of the fluctuation-dissipation theorem (24), I obtain

$$\mathscr{D}=\alpha\bar{\bar{\mathscr{E}}} \tag{42}$$

which would only coincide with Einstein's expression for frequencies satisfying $h\nu\ll T$. An interesting comparison between the two diffusion coefficients can be drawn from the theory of Brownian motion. Let Δ be the momentum due to the irregular radiative interactions transmitted to harmonically bound electrons of frequency ν in a small time interval τ. This counteracts the radiative force $M\gamma v$ opposing the motion. According to the Brownian motion phenomenon, $\overline{\Delta^2}=\mathscr{D}\tau$, expression (42) for the diffusion coefficient in momentum space predicts that the mean square deviation in the momentum fluctuations is

$$\overline{\Delta^2}=\beta\bar{\mathscr{E}}\tau \tag{43}$$

Alternatively, according to the Einstein relation (41), which implicitly implies the equivalence of α and β, I would get

$$\overline{\Delta^2}=\beta\bar{\mathscr{E}}\tau_0 \tag{44}$$

The two expressions differ in their definition of the characteristic times involved: whereas $\tau_0=1/\nu$ is the period of free vibrations, the time interval τ is the mean lifetime of the oscillator prior to absorbing a quantum of energy. The two characteristic times are related by

$$\tau=\tau_0\, e^{h\nu/T} \tag{45}$$

where the exponential factor can be interpreted as the average number of oscillations prior to absorbing a quantum of energy $h\nu$.

I now invert Einstein's procedure: given the fluctuation-dissipation theorem (42), equation (38) is to be solved for the stationary probability density. I thus obtain

$$f^s(v)=\left[\frac{M(1-e^{-h\nu/T})}{2\pi h\nu}\right]^{1/2}\exp\left\{-\frac{1}{2}\frac{Mv^2}{h\nu}(1-e^{-h\nu/T})\right\} \tag{46}$$

In other words, on the strength of the fluctuation-dissipation condition (42) and the condition of a dynamical equilibrium between the radiation damping and the statistical fluctuations produced by the radiation pressure (38), I am led to a modification of the Maxwell velocity distribution (39). It leads to the prediction that twice the average kinetic energy of the oscillator is

$$M\overline{v^2}=\frac{h\nu}{1-e^{-h\nu/T}} \tag{47}$$

which for low frequencies goes over into the equipartition law

$$M\overline{v^2}=T \tag{48}$$

while for high frequencies there results

$$M\overline{v^2}=h\nu \tag{49}$$

This is precisely the zero-point energy in the expression

$$\frac{\bar{\mathscr{E}}}{m}=\frac{h\nu}{e^{h\nu/T}-1}+h\nu \tag{50}$$

proposed by Einstein and Stern (1913) and which is responsible for molecular agitation at zero K.

The pressure per unit volume, given by the well-known kinetic theory expression

$$p = (m/3)M\overline{v^2} \tag{51}$$

would vanish if the average kinetic energy were equated to Planck's expression (19) in the limit as $T \to 0$, while if it were given by (47), I would find

$$p = \tfrac{1}{3}mh\nu = \tfrac{1}{3}\overline{\mathscr{E}} \tag{52}$$

in the same limit. This is the well-known relation for a photon gas, which differs from the usual relation $p = \frac{2}{3}\overline{\mathscr{E}}$ for a gas of material particles.

Finally, the criterion for the valdity of the electrodynamic analysis of the radiation pressure is simply

$$\overline{\Delta^2} \ll M\overline{v^2} \tag{53}$$

Upon inserting (43) and (47) into inequality (53), I get

$$\gamma \ll \nu \tag{54}$$

where I have set $\tau_0 = 1/\nu$, the period of the free vibrations. This expresses the fact that the lifetime $1/\gamma$ of the oscillator is long compared to one period, for otherwise the motion would not even be approximately periodic (Heitler, 1954). The same criterion is obtained when the Einstein relation (44) is used in conjunction with the equipartition law $M\overline{v^2} = T$ in the low-frequency limit. However, for high frequencies or low temperatures it would give a condition involving the temperature and this is foreign to either the classical or quantum theory of natural line breadth caused by damping resulting from the emission of radiation.

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